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A FUNDAMENTAL DISLOCATION SOLUTION FOR AN INFINITE PLATE WITH APPLICATION TO RELATED CRACK PROBLEMS

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Abstract—A fundamental solution for the three-dimensional stress field induced in an infinite plate by an infinitesimal dislocation loop in the interior of the plate is presented here. The solution is derived from the associated Green's function for the same geometry, which is, in turn, found by employing an image method and Muki's formulation [Muki, R. (1960) Asymmetric problems of the theory of elasticity for a semi-infinite solid and a thick plate. In *Progress in Solid Mechanics*, Vol. 1, (eds I. N. Sneddon and R. Hill) Interscience Publishers, New York, pp. 399–439] for an axi-symmetric elastic body. The solution obtained falls naturally into three parts: the first part is singular, and corresponds to the solution for a full space; the second part is regular, and represents the image of the first part to account for the presence of the upper surface of the plate; the third part is also regular, and gives the correction term to maintain the lower surface of the plate free of tractions. The first two terms are expressed in closed form, whilst the third term is expressed in Hankel integral form. Convergence of the integrals is ensured by an asymptotic analysis.

The fundamental dislocation solution found is then employed to analyze the growth of planar cracks in a plate, where the cracks are modelled by a continuous distribution of infinitesimal dislocation loops over the crack faces, i.e., the eigenstrain procedure. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

The analysis of non-uniform advancement of a crack front in a three-dimensional elastic body is a challenging problem, as very few methods which are both accurate and numerically efficient are available. Recent research efforts have focused on the modelling of three-dimensional cracks by a continuous distribution of eigenstrain over the crack faces (Murakami and Nemat-Nasser, 1982; Lee *et al.*, 1987; Dai *et al.*, 1993). In particular, infinitesimal dislocation loops have been employed, so that the problem can be formulated as a set of hypersingular integral equations. These may be solved by using modern numerical methods (Dai *et al.*, 1996). In principle, this technique can be applied to any geometry, but the solution is needed for the stress field produced by an infinitesimal dislocation loop present in the geometry under consideration. Consequently, the technique has hitherto been applied only to some simple geometries which have a closed-form solution, such as an infinite space, a half-space and two-bonded half-spaces. Recently, the technique was also applied to a multilayered half-space by Kuo and Keer (1995). The object of this paper is to present an analytical expression of the fundamental dislocation solution for an infinite plate, and to employ it to analyze some related crack problems.

For simple geometries, such as an infinite space, the fundamental dislocation solution may be obtained by solving the governing equations of elasticity theory, with the specified boundary conditions; see, for example, Burgers (1939) for a treatment of an infinitesimal dislocation loop with an arbitrary Burgers vector, Hanson (1990) for an infinitesimal dislocation loop with its Burgers vector perpendicular to the slip plane, but in an infinite space containing a penny-shaped crack, and Demir *et al.* (1992) for a ring dislocation loop with constant radial Burgers vector. It can be shown by using Volterra's formula (Mura, 1982) that the fundamental dislocation solution is related to the associated Green's function for the same geometry, so that the original problem is reduced to the solution for an infinite plate subject to a point force. The latter solution has been obtained by Benitez and Rosakis (1987) using the Fourier transform method, and by the authors (Dai and Hills, 1995) using a different method, where the spirit of an image method was followed and Muki's formulation (Muki, 1960) for an axi-symmetrical body was then employed to find the images. The object of our re-examining the Green's function solution was twofold: (1) to re-cast the Green's function in such a form that its singularity is explicitly isolated. This is very important when applying the corresponding fundamental dislocation solution to crack problems, because the associated integral equations are hypersingular and an analytical treatment of the singularity is necessitated to evaluate the integrals in a finite part sense (Dai *et al.*, 1996). (2) To find the fundamental dislocation solution. In order to do this, it makes sense to express the very complicated Green's function solution in a more systematic and concise form amenable to subsequent evaluation, which is required in deriving the fundamental dislocation solution itself.

In what follows, we set out to derive the relationship between the fundamental dislocation solution and the associated Green's function. In Section 3 we outline the procedure for finding the Green's function for an infinite plate. This is followed by a presentation of the analytical expression for the corresponding fundamental dislocation solution. As will transpire, the expression splits naturally into three parts: the first part is singular, and corresponds to the solution for a full space; the second part is regular, and represents the image of the first part to account for the presence of the upper surface of the plate; the third part is also regular, and gives the correction term to keep the lower surface of the plate free of tractions. The first two terms, which represent the solution for a half-space, have closed form, whilst the third term is expressed in Hankel integral form. In Section 5 we employ the fundamental dislocation solution to analyze some related crack problems for a plate, by modelling the cracks as a continuous distribution of infinitesimal dislocation loops over the crack faces. As an illustration of the technique, a growth analysis of planar surface cracks under fatigue will be conducted.

2. FUNDAMENTAL RELATIONSHIPS

From dislocation theory, the displacement field in an elastic body produced by a Volterra dislocation of Burgers vector b_j, with slip plane S and a unit normal n_k, is given by (Mura, 1982)

$$u_i(\mathbf{x}) = \int_S C_{jkmn} \frac{\partial G_{mi}(\mathbf{x}, \mathbf{x}')}{\partial x'_n} b_j n_k \, \mathrm{d}S, \tag{1}$$

where C_{jkmn} is the tensor of elastic constants, and $G_{mi}(\mathbf{x}, \mathbf{x}')$ is the Green's function for the geometry under consideration, representing the displacement in the i-direction at a point $\mathbf{x}$ due to a unit point force in the m-direction at the point $\mathbf{x}'$. The associated stress field is given by Hooke's law, i.e.,

$$\sigma_{il}(\mathbf{x}) = C_{ilpq} u_{p,q}(\mathbf{x}) = \int_S H_{iljk}(\mathbf{x}, \mathbf{x}') b_j n_k \, \mathrm{d}S, \tag{2}$$

where

$$H_{iljk}(\mathbf{x}, \mathbf{x}') = C_{ilpq} C_{jkmn} \frac{\partial^2 G_{mp}(\mathbf{x}, \mathbf{x}')}{\partial x_q \partial x'_n} = C_{jkmn} \frac{\partial T_{ilm}(\mathbf{x}, \mathbf{x}')}{\partial x'_n}. \tag{3}$$

Here, $T_{ilm}(\mathbf{x}, \mathbf{x}')$ is the stress field produced by the unit point force, or the stress Green's function. Now, if we let the slip plane S shrink to become infinitely small, i.e., $S \to \delta S$, we obtain an *infinitesimal* dislocation loop (Eshelby, 1960), which is defined by its strength

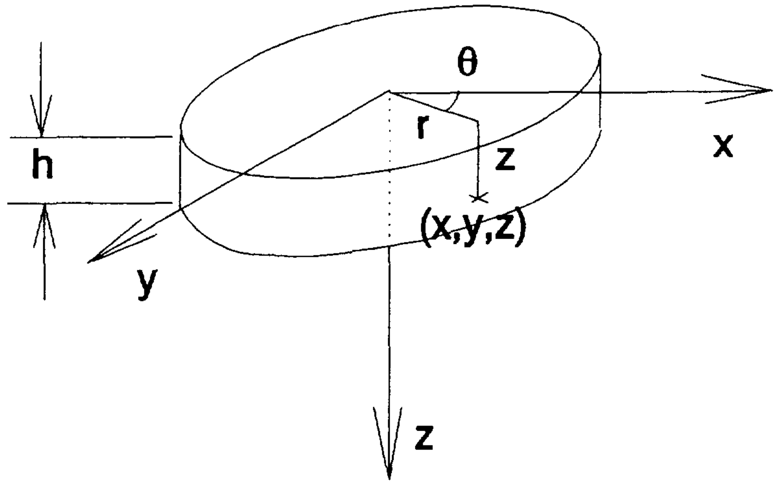

Fig. 1. Cartesian coordinate set, fixed on the upper surface of the plate, with the z-axis pointing towards the interior of the plate.

tensor $s_{jk} = b_j n_k \delta S$. Consequently, eqn (2) becomes

$$\sigma_{il}(\mathbf{x}) = H_{iljk}(\mathbf{x}, \mathbf{x}')s_{jk}. \tag{4}$$

Thus, it can be seen that physically $H_{iljk}(\mathbf{x}, \mathbf{x}')$ represents the stress field produced by an infinitesimal dislocation loop with unit strength tensor, i.e., when its jk-th component is unity and all other components are zero. Further, we have

$$H_{iljk}(\mathbf{x}, \mathbf{x}') = \mu \left[\frac{\partial T_{ilj}(\mathbf{x}, \mathbf{x}')}{\partial x'_k} + \frac{\partial T_{ilk}(\mathbf{x}, \mathbf{x}')}{\partial x'_j} + \frac{2\nu}{1-2\nu} \frac{\partial T_{ilm}(\mathbf{x}, \mathbf{x}')}{\partial x'_m} \delta_{jk} \right], \tag{5}$$

for an isotropic elastic body, where μ and ν are the shear modulus and Poisson's ratio of the material.

3. GREEN'S FUNCTION

Equation (3) shows that, for the same geometry, the fundamental dislocation solution, $H_{iljk}(\mathbf{x}, \mathbf{x}')$, is related to the associated Green's function, $G_{mi}(\mathbf{x}, \mathbf{x}')$, or the corresponding stress field, $T_{ilm}(\mathbf{x}, \mathbf{x}')$. Thus, the former can be obtained by differentiating the latter with respect to the source point $\mathbf{x}'$. In order to find the fundamental dislocation solution, we may first find the associated Green's function.

The stress Green's function for an infinite plate subject to a unit point force at an arbitrary point was documented in detail in a recent report by the authors (Dai and Hills, 1995). The results are extremely lengthy, and here we give only a brief description of the solution procedure. In doing so, a Cartesian coordinate set is chosen to fix on the upper surface of the plate, with the z axis perpendicular to this surface and pointing inwards to the interior of the plate, as shown in Fig. 1. All coordinates are normalized with respect to the thickness of the plate, h. Employing the 'image' method, we split the displacement and stress fields into two parts,

$$G_{mi}(\mathbf{x}, \mathbf{x}') = G^s_{mi}(\mathbf{x}, \mathbf{x}') + G^c_{mi}(\mathbf{x}, \mathbf{x}') \tag{6}$$

and

$$T_{ijm}(\mathbf{x},\mathbf{x}') = T^s_{ijm}(\mathbf{x},\mathbf{x}') + T^c_{ijm}(\mathbf{x},\mathbf{x}'), \tag{7}$$

where G^s_{mi} and T^s_{ijm} are the Green's function (Kelvin's solution) for an infinite space (see Love, 1927, p. 183), and G^c_{mi} and T^c_{ijm} are the images of G^s_{mi} and T^s_{ijm}, or the correction terms to account for the presence of the two free surfaces of the plate. It is evident from the superposition principle that G^c_{mi} and T^c_{ijm} must satisfy the homogeneous equations of equilibrium and Hooke's law. Further, they are required to satisfy the boundary condition

$$T^c_{3jm}(\mathbf{x},\mathbf{x}') = -T^s_{3jm}(\mathbf{x},\mathbf{x}'), \quad z = 0, 1, \tag{8}$$

to ensure that the two surfaces of the plate are tractions free.

Following Muki's formulation for an axi-symmetric elastic body (Sneddon, 1949; Muki, 1960), we may express the solution of the homogeneous equations of equilibrium by a biharmonic function ϕ and a harmonic function ψ. Further, we may expand these two functions using a Fourier's series with respect to the circumferential coordinate θ (in a cylindrical polar coordinate system), and then represent the coefficient of each harmonic component (itself a function of the coordinates ρ and z) by Hankel's transformation with respect to the radial coordinate ρ, so that we can deduce the form of the functions, ϕ and ψ, leaving only some constants undetermined. By doing so, we can write the expressions for T^c_{ijm} in the forms (Dai and Hills, 1995),

$$\frac{T^c_{\rho\rho m}}{2\mu} = \sum_{n=0}^{\infty}\left\{-\frac{n+1}{2\rho h}W^m_{n+1} - \frac{n-1}{2\rho h}V^m_{n-1} + \frac{1}{h^3}\int_0^{\infty}\{[A^m_n + (1+2\nu+\alpha z)B^m_n]e^{\alpha z}\right.$$
$$\left. + [-C^m_n + (1+2\nu-\alpha z)D^m_n]e^{-\alpha z}\}\alpha^2 J_n(\alpha\rho)\,\mathrm{d}\alpha\right\}\cos n\theta$$

$$\frac{T^c_{\theta\theta m}}{2\mu} = \sum_{n=0}^{\infty}\left\{\frac{n+1}{2\rho h}W^m_{n+1} + \frac{n-1}{2\rho h}V^m_{n-1} + \frac{2\nu}{h^3}\int_0^{\infty}[B^m_n e^{\alpha z} + D^m_n e^{-\alpha z}]\alpha^2 J_n(\alpha\rho)\,\mathrm{d}\alpha\right\}\cos n\theta$$

$$\frac{T^c_{zzm}}{2\mu} = \sum_{n=0}^{\infty}\frac{1}{h^3}\left\{\int_0^{\infty}\{[-A^m_n + (1-2\nu-\alpha z)\beta^m_n]e^{\alpha z}\right.$$
$$\left. + [C^m_n + (1-2\nu+\alpha z)D^m_n]e^{-\alpha z}\}\alpha^2 J_n(\alpha\rho)\,\mathrm{d}\alpha\right\}\cos n\theta$$

$$\frac{T^c_{\rho\theta m}}{2\mu} = \sum_{n=0}^{\infty}\left\{-\frac{n+1}{2\rho h}W^m_{n+1} + \frac{n-1}{2\rho h}V^m_{n-1} + \frac{1}{h^3}\int_0^{\infty}[E^m_n e^{\alpha z} + F^m_n e^{-\alpha z}]\alpha^2 J_n(\alpha\rho)\,\mathrm{d}\alpha\right\}\sin n\theta$$

$$\frac{T^c_{\theta zm}}{2\mu} = \sum_{n=0}^{\infty}\frac{1}{2}[R^m_{n+1} + S^m_{n-1}]\sin n\theta$$

$$\frac{T^c_{z\rho m}}{2\mu} = \sum_{n=0}^{\infty}\frac{1}{2}[R^m_{n+1} - S^m_{n-1}]\cos n\theta, \tag{9}$$

where $\rho = \sqrt{(x-x')^2+(y-y')^2}$ is the radial distance from the field point (x, y, z) to the source point (x', y', z'), and A^m_n, B^m_n, C^m_n, D^m_n, E^m_n and F^m_n are constants to be determined by boundary condition (8), whilst functions W^m_{n+1}, V^m_{n-1}, R^m_{n+1} and S^m_{n-1} are defined by

$$
\begin{aligned}
W^m_{n+1}(\rho,z) &= \frac{1}{h^2}\int_0^\infty \{[A^m_n+(1+\alpha z)B^m_n+2E^m_n]e^{\alpha z} \\
&\quad +[-C^m_n+(1-\alpha z)D^m_n+2F^m_n]e^{-\alpha z}\}\alpha J_{n+1}(\alpha\rho)\,\mathrm{d}\alpha \\
V^m_{n-1}(\rho,z) &= \frac{1}{h^2}\int_0^\infty \{[A^m_n+(1+\alpha z)B^m_n-2E^m_n]e^{\alpha z} \\
&\quad +[-C^m_n+(1-\alpha z)D^m_n-2F^m_n]e^{-\alpha z}\}\alpha J_{n-1}(\alpha\rho)\,\mathrm{d}\alpha \\
R^m_{n+1}(\rho,z) &= \frac{1}{h^3}\int_0^\infty \{[A^m_n+(2\nu+\alpha z)B^m_n+E^m_n]e^{\alpha z} \\
&\quad +[C^m_n+(-2\nu+\alpha z)D^m_n-F^m_n]e^{-\alpha z}\}\alpha^2 J_{n+1}(\alpha\rho)\,\mathrm{d}\alpha \\
S^m_{n-1}(\rho,z) &= \frac{1}{h^3}\int_0^\infty \{[A^m_n+(2\nu+\alpha z)B^m_n-E^m_n]c^{\alpha z} \\
&\quad +[C^m_n+(-2\nu+\alpha z)D^m_n+F^m_n]e^{-\alpha z}\}\alpha^2 J_{n-1}(\alpha\rho)\,\mathrm{d}\alpha.
\end{aligned} \tag{10}
$$

In view of the boundary condition (8) and the mathematical structure of Kelvin's solution, we may set $E^m_n = F^m_n \equiv 0$ when the unit point force is directed along z axis ($m = 3$), and retain the first term only of the series in eqn (9). The four remaining constants, A^3_0, B^3_0, C^3_0 and D^3_0 can be readily found by imposing boundary condition (8) (four equations in a cylindrical polar coordinate set). Further, when the unit point force is parallel with the plane of the plate, say along the x axis ($m = 1$), we may retain the second term of the series only. The six remaining constants A^1_1, B^1_1, C^1_1, D^1_1, E^1_1 and F^1_1 can also be found from boundary condition (8) (six equations in this case). In both cases, the constants A^m, B^m, C^m and D^m (the subscripts 0 to 1 may be omitted without causing confusion) are given by

$$
\begin{aligned}
A^m &= \frac{K_d h}{\alpha D(\alpha)}\sum_{i=1}^4 P^m_{1i}(\alpha,z')e^{-z'_i\alpha} \\
B^m &= \frac{K_d h}{\alpha D(\alpha)}\sum_{i=1}^4 P^m_{2i}(\alpha,z')e^{-z'_i\alpha} \\
C^m &= \frac{K_d h}{\alpha D(\alpha)}\sum_{i=1}^4 P^m_{3i}(\alpha,z')e^{-z'_i\alpha} - \frac{K_d h}{\alpha}P^m_{34}(\alpha,z')e^{-z'\alpha} \\
D^m &= \frac{K_d h}{\alpha D(\alpha)}\sum_{i=1}^4 P^m_{4i}(\alpha,z')e^{-z'_i\alpha} - \frac{K_d h}{\alpha}P^m_{44}(\alpha,z')e^{-z'\alpha},
\end{aligned} \tag{11}
$$

where $K_d = 1/[16\pi\mu(1-\nu)]$ is a material constant, $D(\alpha) = 1-2(1+2\alpha^2)e^{-2\alpha}+e^{-4\alpha}$ is an analytical function and is of order α^4 when α approaches zero, $z'_i = 2[(i+1)/2]+(-1)^i z'$ ($i = 1,2,3,4$) (here $[\cdot]$ denotes the truncation function), and $P^m_{ji}(\alpha,z')$ ($j,i = 1,\ldots,4$) are polynomials in α, which are given in Appendix A. Constants E^1_1 and F^1_1 are given by

$$
\begin{aligned}
E^1_1 &= 2(1-\nu)\frac{K_d h}{\alpha H(\alpha)}[e^{-(2-z')\alpha}+e^{-(2+z')\alpha}] \\
F^1_1 &= 2(1-\nu)\frac{K_d h}{\alpha H(\alpha)}[e^{-(2-z')\alpha}+e^{-(2+z')\alpha}]+2(1-\nu)\frac{K_d h}{\alpha}e^{-z'\alpha},
\end{aligned} \tag{12}
$$

where $H(\alpha) = 1+e^{-2\alpha}$ is also an analytical function and is of order α when α approaches zero.

We now turn to the determination of the stress fields. This can be done by substituting (11) and (12) back into (9) and (10). Interestingly, we find that Mindlin's solution for a half-space is recovered from part of the integral expressions in (9) which are associated with the second terms of constants C^m, D^m and F_1^1, i.e., those terms without denominator $D(\alpha)$ or $H(\alpha)$. In a Cartesian coordinate system, the final expressions for the stress field, T^c_{ijm}, are,

$$
\begin{aligned}
T^c_{\gamma\gamma\gamma} &= T^h_{\gamma\gamma\gamma} + \frac{K_s}{h^2}\frac{\rho_\gamma}{\rho}\left[L_1\left(S_0+\frac{1}{4}U_-\right)+\frac{1}{4}\left(3-4\frac{\rho_\gamma^2}{\rho^2}\right)L_3(U_+)\right] \\
T^c_{\beta\beta\gamma} &= T^h_{\beta\beta\gamma} + \frac{K_s}{h^2}\frac{\rho_\gamma}{\rho}\left[L_1\left(S_0-\frac{1}{4}U_-\right)-\frac{1}{4}\left(3-4\frac{\rho_\gamma^2}{\rho^2}\right)L_3(U_+)\right] \\
T^c_{12\gamma} &= T^h_{12\gamma} + \frac{K_s}{4h^2}\frac{\rho_{3-\gamma}}{\rho}\left[L_1(U_-)+\left(1-4\frac{\rho_\gamma^2}{\rho^2}\right)L_3(U_+)\right] \\
T^c_{\gamma3\gamma} &= T^h_{\gamma3\gamma} - \frac{K_s}{2h^2}\left[L_0(S_-)+\left(1-2\frac{\rho_\gamma^2}{\rho^2}\right)L_2(S_+)\right] \\
T^c_{\beta3\gamma} &= T^h_{\beta3\gamma} + \frac{K_s}{h^2}\frac{\rho_\beta\rho_\gamma}{\rho^2}L_2(S_+) \\
T^c_{33\gamma} &= T^h_{33\gamma} + \frac{K_s}{h^2}\frac{\rho_\gamma}{\rho}L_1(S_3) \\
T^c_{\gamma\gamma3} &= T^h_{\gamma\gamma3} + \frac{K_s}{h^2}\left[L_0(S_0^3)+\frac{1}{2}\left(1-2\frac{\rho_\gamma^2}{\rho^2}\right)L_2(U_1^3)\right] \\
T^c_{123} &= T^h_{123} - \frac{K_s}{h^2}\frac{\rho_1\rho_2}{\rho^2}L_2(U_1^3) \\
T^c_{\gamma33} &= T^h_{\gamma33} + \frac{K_s}{h^2}\frac{\rho_\gamma}{\rho}L_1(S_4^3) \\
T^c_{333} &= T^h_{333} + \frac{K_s}{h^2}L_0(S_3^3),
\end{aligned}
\tag{13}
$$

where $K_s = 1/[8\pi(1-\nu)]$, $\rho_\gamma = x_\gamma - x'_\gamma$, γ, $\beta = 1, 2$, but $\gamma \neq \beta$, and no summation is implied by a repeated subscript. Further, T^h_{ijm} are the correction terms within Mindlin's solution which account for the free surface of a half-space (Mindlin, 1936), functions S_0, $U_-, \ldots$, S_3^3 are given by the product of polynomials P^m_{ji} and exponent functions $e^{-(z'_i-z)\alpha}$ or $e^{-(z'_i+z)\alpha}$ (see Appendix A), and $L_n(F)$ $(n = 0, 1, 2, 3)$ is a family of integrals, given by

$$
L_n(F) = \int_0^\infty \frac{F(\alpha, z, z')}{D(\alpha)}\alpha J_n(\alpha\rho)\,\mathrm{d}\alpha. \tag{14}
$$

Here $J_m(\alpha\rho)$ is the mth order Bessel function of the first kind, and $F(\alpha, z, z')$ represents one of the functions, S_0, $U_-, \ldots, S_3^3$.

Finally, using eqns (7) and (13), we can split the Green's function, $T_{ijm}(\mathbf{x},\mathbf{x}')$, for an infinite plate into three parts

$$T_{ilm}(\mathbf{x},\mathbf{x}') = T^{s}_{ilm}(\mathbf{x},\mathbf{x}')+T^{h}_{ilm}(\mathbf{x},\mathbf{x}')+T^{p}_{ilm}(\mathbf{x},\mathbf{x}') \tag{15}$$

where $T^{p}_{ilm}(\mathbf{x},\mathbf{x}')$ represent the integral terms in eqn (13).

It follows from definition (14) that the integral family L_m may be divergent as the denominator $D(\alpha)$ approaches α^4 at $\alpha = 0$. All integral expressions included in eqn (13) must therefore be examined to ensure their convergence. Asymptotic expansion analysis reveals that the integral expressions for $T^{p}_{\gamma\gamma 3}$ are divergent, i.e., the in-plane normal stresses due to a unit point force perpendicular to the plane of the plate are singular, but the remaining expressions are convergent. To remove divergence of the integrals, we have to add the following solution (Dai and Hills, 1995),

$$T^{a}_{113} = T^{a}_{223} = 16A\frac{K_s}{h^2}(1-\nu^2)(1-2z), \tag{16}$$

where A is an appropriate unbounded constant. Clearly, eqn (16) represents the stress state induced by pure bending of the plate. The underlying reason for having to superpose this term is that an infinite plate can undergo an unbounded pure bending deformation under the action of normal point force, with the boundary conditions we have employed. Hence a similar deformation state but with opposite sign must be superposed to cancel this unwanted deformation mode. Nevertheless, these additional terms are irrelevant for our purpose, because they are independent of the position of the source point, $\mathbf{x}'$, and consequently make no contribution to the fundamental dislocation solution (c.f. eqns (3) or (5)). Equation (13) is therefore expected to provide a convergent solution after performing the appropriate algebraic and differential operations given by eqn (5).

4. FUNDAMENTAL DISLOCATION SOLUTION

By using eqns (5), (13) and (15) and the following relations,

$$L_{n-1}(\alpha F)+L_{n+1}(\alpha F) = \frac{2n}{\rho}L_n(F) \tag{17}$$

$$L_{n-1}(\alpha F)-L_{n+1}(\alpha F) = \frac{2\,\mathrm{d}L_n(F)}{\mathrm{d}\rho}, \tag{18}$$

we are able to derive expressions for the fundamental dislocation solution. As with eqn (15), we split the solution into three parts,

$$H_{iljk}(\mathbf{x},\mathbf{x}') = H^{s}_{iljk}(\mathbf{x},\mathbf{x}')+H^{h}_{iljk}(\mathbf{x},\mathbf{x}')+H^{p}_{iljk}(\mathbf{x},\mathbf{x}'), \tag{19}$$

where $H^{s}_{iljk}(\mathbf{x},\mathbf{x}')$, $H^{h}_{iljk}(\mathbf{x},\mathbf{x}')$ and $H^{p}_{iljk}(\mathbf{x},\mathbf{x}')$ are determined from $T^{s}_{ilm}(\mathbf{x},\mathbf{x}')$, $T^{h}_{ilm}(\mathbf{x},\mathbf{x}')$ and $T^{p}_{ilm}(\mathbf{x},\mathbf{x}')$, respectively, by using eqn (5). In other words, the first term, $H^{s}_{iljk}(\mathbf{x},\mathbf{x}')$, gives the fundamental dislocation solutions for an infinite space, whilst the first two terms, $H^{s}_{iljk}(\mathbf{x},\mathbf{x}')+H^{h}_{iljk}(\mathbf{x},\mathbf{x}')$, represent the fundamental dislocation solution for a half-space. These solutions have been given in different forms by Burgers (1939), Nabarro (1951), Eshelby (1961), Bacon and Groves (1970). Here we give the expression for the plate correction, $H^{p}_{iljk}(\mathbf{x},\mathbf{x}')$, only. It is evident from eqn (3) that $H_{iljk}(\mathbf{x},\mathbf{x}') = H_{lijk}(\mathbf{x},\mathbf{x}') = H_{ilkj}(\mathbf{x},\mathbf{x}')$, i.e., the fundamental dislocation solution has, at most, 36 independent components for an arbitrary geometry. These independent components can be divided into six sets, and in terms of the strength tensor, s_{jk}, of the infinitesimal dislocation loop they are:

Dislocation loop s_{11} and s_{22}

$$H^p_{\gamma\gamma\gamma\gamma} = -\frac{K_h}{2h^3}\left[L_0\left(\alpha S_0 + \frac{\alpha}{4}U_-\right) + \left(1-2\frac{\rho_\gamma^2}{\rho^2}\right)L_2\left(\alpha S_0 + \frac{\alpha}{2}U_1\right) + \frac{1}{4}\left(1-8\frac{\rho_1^2\rho_2^2}{\rho^4}\right)L_4(\alpha U_+) + h_{\gamma\gamma}\right]$$

$$H^p_{\beta\beta\gamma\gamma} = -\frac{K_h}{2h^3}\left[L_0\left(\alpha S_0 - \frac{\alpha}{4}U_-\right) + \left(1-2\frac{\rho_\gamma^2}{\rho^2}\right)L_2\left(\alpha S_0 - \frac{\alpha}{2}U_1\right) - \frac{1}{4}\left(1-8\frac{\rho_1^2\rho_2^2}{\rho^4}\right)L_4(\alpha U_+) + h_{\beta\beta}\right]$$

$$H^p_{12\gamma\gamma} = \frac{K_h}{2h^3}\frac{\rho_1\rho_2}{\rho^2}\left[L_2(\alpha U_1) + \left(1-2\frac{\rho_\gamma^2}{\rho^2}\right)L_4(\alpha U_+) + h_{12}\right]$$

$$H^p_{\gamma 3\gamma\gamma} = -\frac{K_h}{2h^3}\frac{\rho_\gamma}{\rho}\left[L_1(\alpha S_-) + \frac{1}{2}L_1(\alpha S_+) + \frac{1}{2}\left(3-4\frac{\rho_\gamma^2}{\rho^2}\right)L_3(\alpha S_+) + h_{\gamma 3}\right]$$

$$H^p_{\beta 3\gamma\gamma} = -\frac{K_h}{2h^3}\frac{\rho_\beta}{\rho}\left[\frac{1}{2}L_1(\alpha S_+) + \frac{1}{2}\left(1-4\frac{\rho_\gamma^2}{\rho^2}\right)L_3(\alpha S_+) + h_{\beta 3}\right]$$

$$H^p_{33\gamma\gamma} = -\frac{K_h}{2h^3}\left[L_0(\alpha S_3) + \left(1-2\frac{\rho_\gamma^2}{\rho^2}\right)L_2(\alpha S_3) + h_{33}\right]. \tag{20}$$

Dislocation loop s_{12}

$$H^p_{\gamma\gamma 12} = \frac{K_h}{2h^3}\frac{\rho_1\rho_2}{\rho^2}\left[2L_2(\alpha S_0) + \left(1-2\frac{\rho_\gamma^2}{\rho^2}\right)L_4(\alpha U_+)\right]$$

$$H^p_{1212} = -\frac{K_h}{8h^3}\left[L_0(\alpha U_-) - \left(1-8\frac{\rho_1^2\rho_2^2}{\rho^4}\right)L_4(\alpha U_+)\right]$$

$$H^p_{\gamma 312} = -\frac{K_h}{4h^3}\frac{\rho_{3-\gamma}}{\rho}\left[L_1(\alpha S_-) + \left(1-4\frac{\rho_\gamma^2}{\rho^2}\right)L_3(\alpha S_+)\right]$$

$$H^p_{3312} = \frac{K_h}{h^3}\frac{\rho_1\rho_2}{\rho^2}L_2(\alpha S_3). \tag{21}$$

Dislocation loop s_{13} and s_{23}

$$H^p_{\gamma\gamma\gamma 3} = \frac{K_h}{2h^3}\frac{\rho_\gamma}{\rho}\left[L_1(S_{0,z'} + \alpha S_0^3) + \frac{1}{4}L_1(U_{-,z'} + \alpha U_1^3) + \frac{1}{4}\left(3-4\frac{\rho_\gamma^2}{\rho^2}\right)L_3(U_{+,z'} + \alpha U_1^3)\right]$$

$$H^p_{\beta\beta\gamma 3} = \frac{K_h}{2h^3}\frac{\rho_\gamma}{\rho}\left[L_1(S_{0,z'}+\alpha S_0^3) - \frac{1}{4}L_1(U_{-,z'}+\alpha U_1^3) - \frac{1}{4}\left(3-4\frac{\rho_\gamma^2}{\rho^2}\right)L_3(U_{+,z'}+\alpha U_1^3)\right]$$

$$H^p_{12\gamma 3} = \frac{K_h}{8h^3}\frac{\rho_{3-\gamma}}{\rho}\left[L_1(U_{-,z'}+\alpha U_1^3) + \left(1-4\frac{\rho_\gamma^2}{\rho^2}\right)L_3(U_{+,z'}+\alpha U_1^3)\right]$$

$$H^p_{\gamma 3\gamma 3} = -\frac{K_h}{4h^3}\left[L_0(S_{-,z'}+S_4^3) + \left(1-2\frac{\rho_\gamma^2}{\rho^2}\right)L_2(S_{+,z'}+\alpha S_4^3)\right]$$

$$H^p_{\gamma 3\beta 3} = \frac{K_h}{2h^3}\frac{\rho_\gamma\rho_\beta}{\rho^2}L_2(S_{+,z'}+\alpha S_4^3)$$

$$H^p_{33\gamma 3} = \frac{K_h}{2h^3}\frac{\rho_\gamma}{\rho}L_1(S_{3,z'}+\alpha S_3^3). \qquad (22)$$

Dislocation loop s_{33}

$$H^p_{\gamma\gamma 33} = \frac{K_h}{2h^3}\left[2L_0(S^3_{0,z'}) + \left(1-2\frac{\rho_\gamma^2}{\rho^2}\right)L_2(U^3_{1,z'}) - h_{\gamma\gamma}\right]$$

$$H^p_{1233} = -\frac{K_h}{2h^3}\frac{\rho_1\rho_2}{\rho^2}[L_2(U^3_{1,z'}) - h_{12}]$$

$$H^p_{\gamma 333} = \frac{K_h}{2h^3}\frac{\rho_\gamma}{\rho}[2L_1(S^3_{4,z'}) - h_{\gamma 3}]$$

$$H^p_{3333} = \frac{K_h}{2h^3}[2L_0(S^3_{3,z'}) - h_{33}]. \qquad (23)$$

In these expressions, γ, $\beta = 1, 2$, but $\gamma \neq \beta$, and no summation is implied in any case. Further, $K_h = \mu/[4\pi(1-\nu)]$ is a material constant, $F_{,z'}$ denotes the derivatives of the function $F(\alpha, z, z')$ specified with respect to z', and h_{ij} is given by

$$h_{\gamma\gamma} = \frac{\nu}{1-2\nu}\left[2L_0(\alpha S_0 - S^3_{0,z'}) + \left(1-2\frac{\rho_\gamma^2}{\rho^2}\right)L_2(\alpha U_1 - U^3_{1,z'})\right]$$

$$h_{12} = \frac{2\nu}{1-2\nu}L_2(\alpha U_1 - U^3_{1,z'})$$

$$h_{\gamma 3} = \frac{2\nu}{1-2\nu}L_1(\alpha S_4 - S^3_{4,z'})$$

$$h_{33} = \frac{2\nu}{1-2\nu}L_0(\alpha S_3 - S^3_{3,z'}). \qquad (24)$$

So far, we have found a formal fundamental dislocation solution for an infinite plate. The remaining task is to examine the convergence of all integrals involved. This is essential in order to ensure that the solution developed is physically acceptable. Detailed asymptotic analysis shows that all integrands appearing in eqns (20)–(24) are of order α^m ($m \geqslant 0$) when $\alpha \to 0$, and decay in an exponential form when $\alpha \to \infty$. All integrals are therefore convergent, even though the original integral expressions for $T^p_{\gamma\gamma 3}(\mathbf{x}, \mathbf{x}')$ are divergent; this is in consistent with our earlier analysis.

5. RELATED CRACK PROBLEMS

We now turn to crack problems. The fundamental dislocation solution just found is employed as the influence function in modelling the cracks by a continuous distribution of infinitesimal dislocation loops. First, we give a brief review of the formulation.

5.1. *Integral equation formulation*

Suppose that an elastic body containing an arbitrary crack, S, is subject to a remote loading, σ_{ij}^{∞}. The crack faces will, in general, experience three components of relative displacement. For each pair of infinitesimal crack face elements, the corresponding relative displacement vector and the normal of the crack element can now be characterized by an infinitesimal dislocation loop, by specifying its Burgers vector, b_j, and orientation, n_k. As a result, the presence of the cracks in an elastic body can be modelled by a continuous distribution of infinitesimal dislocation loops over the crack faces. The disturbed stresses, σ_{il}^{d}, associated with the presence of the crack, are now given by the stress field produced by the distributed dislocation loops, which are, from eqn (4),

$$\sigma_{il}^{d}(\mathbf{x}) = \int_S H_{iljk}(\mathbf{x},\mathbf{x}')b_j(\mathbf{x}')n_k(\mathbf{x}')\,\mathrm{d}S. \tag{25}$$

The total stress in the elastic body is the sum of the undisturbed and corrective stresses, i.e., $\sigma_{il}(\mathbf{x}) = \sigma_{il}^{0}(\mathbf{x})+\sigma_{il}^{d}(\mathbf{x})$, where $\sigma_{il}^{0}(\mathbf{x})$ are the undisturbed stresses induced by the external loading in the absence of the crack. Hence, the requirement that the crack faces be traction free gives the following integral equations,

$$\int_S K_{ij}(\mathbf{x},\mathbf{x}')b_j(\mathbf{x}')\,\mathrm{d}S = -t_i^{0}(\mathbf{x}) \quad \mathbf{x}\in S \tag{26}$$

where $K_{ij}(\mathbf{x},\mathbf{x}') = H_{iljk}(\mathbf{x},\mathbf{x}')n_l(\mathbf{x})n_k(\mathbf{x}')$ are given kernel functions, and $t_i^{0}(\mathbf{x}) = \sigma_{il}^{0}(\mathbf{x})n_l(\mathbf{x})$ are the known undisturbed tractions over the crack faces.

Integral eqns (26) provide a starting point for the solution of crack problems. In principle, the integral equations may be solved by a numerical method; divide S into a number of elements, and approximate $b_j(\mathbf{x})$ within each element by an interpolation or shape function, incorporating an appropriate weight function for crack front elements to enforce the correct behaviour of the relative displacements near the crack front. By doing so, the integral equations are then reduced to a set of algebraic equations which can be readily inverted. It must be stressed, however, that the kernel functions, $K_{ij}(\mathbf{x},\mathbf{x}')$, have a singularity of order three, $O(1/r^3)$, when r approaches zero. The associated integrals are therefore hypersingular and exist only in the finite part sense, which need special analytical treatment in a numerical solution. For the special case where the crack face lies in a plane, an efficient numerical solution procedure has been proposed by Dai *et al.* (1996), where the hyper-singular integrals were converted into contour integrals (for which closed-form solutions are available for a polygonal element) and a weakly singular integral. This procedure will be employed in the following analysis.

5.2. *Analysis of crack growth*

Using fundamental dislocation solution (19) and integral eqns (26), we are able to analyze crack problems in an infinite plate. In particular, we intend here to model the growth of planar cracks under fatigue loading. To render the problem tractable, we assume that the crack plane is perpendicular to the surfaces of the plate, and that only opening mode loading, e.g., in plane tension or bending, is involved, so that the crack will remain in the same plane during growth. As a demonstration, only tensile remote traction is considered here. The solution is found by a step-by-step algorithm; first, we use our solution procedure (Dai *et al.*, 1996) to solve integral eqns (26) for an initial crack shape, and extract the stress intensity factors for a set of nodes distributed along the crack front; we then use Paris law to predict a new crack front position, which is defined by the new positions of

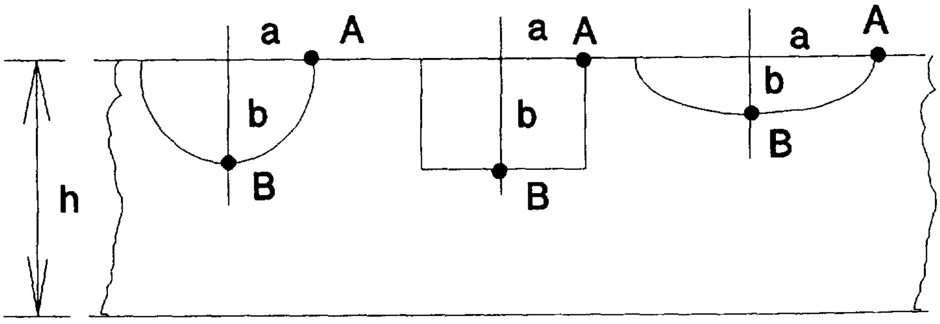

Fig. 2. Three assumed forms of initial crack shape; semi-circular, rectangular and semi-elliptical.

the crack front nodes, and remesh the new crack faces by a dynamic process. Once this has been done, we use integral eqns (26) again to find a new solution. This process is continued step by step, until the cracks approach the backface of the plate.

Three kinds of initial crack shape, as shown in Fig. 2, are considered here. As a check of the accuracy of the present solutions, the stress intensity factors obtained for these initial crack shapes are compared, where appropriate, with some well-established solutions. Table 1 shows a comparison of the results with Newman and Raju's solution (Newman and Raju, 1981) for semi-circular and semi-elliptical cracks. It is found that, for both cracks, the relative errors between the two solutions are less than 2% along the whole crack front, save the points breaking the surface ($\theta = 0°$). The larger errors occurring at the surface are attributed to the change of the stress singularity near the surface, where the order of singularity is less than 0.5 (Benthem, 1977) and is not correctly modelled by either technique. As no appropriate solutions are available for a rectangular crack, we have to compare the results with the corresponding solutions for a half-space (h tends to infinity). Table 2 shows the stress intensity factors at points A (surface) and B (bottom) of the crack in both a plate ($b/h = 0.2$) and a half-space ($b/h = 0$). The corresponding solutions for a semi-circular crack are also included in Table 2, so that the effect of the finite thickness of the plate on the stress intensity factors can be estimated, as very accurate solutions are available for this geometry. It is observed that the increase of the stress intensities due to the presence of the backface of the plate is, as expected, of the same order for both crack shapes. In fact, our algorithm treats all cracks as being of arbitrary shape, and the accuracy of the solution is therefore expected to be independent of the crack shape.

Table 1. Comparison of stress intensity factor for semi-elliptical cracks ($F_I = K_I/\sigma^0\sqrt{\pi b}$, $\nu = 0.3$)

a/b ($b/h = 0.2$)	θ (deg.)	Present solution	Newman and Raju (1981)	Error (%)
2.0	00.00	0.749	0.725	3.3
	05.68	0.722	0.718	0.6
	11.70	0.747	0.743	0.5
	18.47	0.789	0.783	0.8
	26.57	0.834	0.826	0.9
	36.81	0.872	0.864	0.9
	50.36	0.897	0.894	0.3
	68.31	0.910	0.913	−0.3
	90.00	0.914	0.920	−0.7
1.0	00.00	0.764	0.744	2.6
	11.25	0.730	0.717	1.8
	22.50	0.705	0.697	1.1
	33.75	0.690	0.683	1.0
	45.00	0.681	0.674	1.0
	56.25	0.676	0.670	0.9
	67.50	0.672	0.668	0.6
	78.75	0.670	0.668	0.3
	90.00	0.670	0.668	0.3

Table 2. Comparison of stress intensity factor for a plate and a half-space ($F_I = K_I/\sigma^0 \sqrt{\pi b}$, $\nu = 0.3$)

Crack shape	b/h	F_A	F_B
Rectangle	0.2	0.884	0.796
	0.0	0.873	0.789 (0.788†)
Semi-circle	0.2	0.764	0.670
	0.0	0.757	0.665 (0.666‡)

† : Murakami (1992).
‡ : Murakami (1982), Lee *et al.* (1987).

Figure 3 shows the evolution of the crack front by fatigue for the three crack shapes considered. It is evident that the initial growth of the cracks takes a quite different pattern for different initial shapes. This is explained by the fact that the distribution of stress intensity factor along the crack front depends very much on the crack shape, as shown in

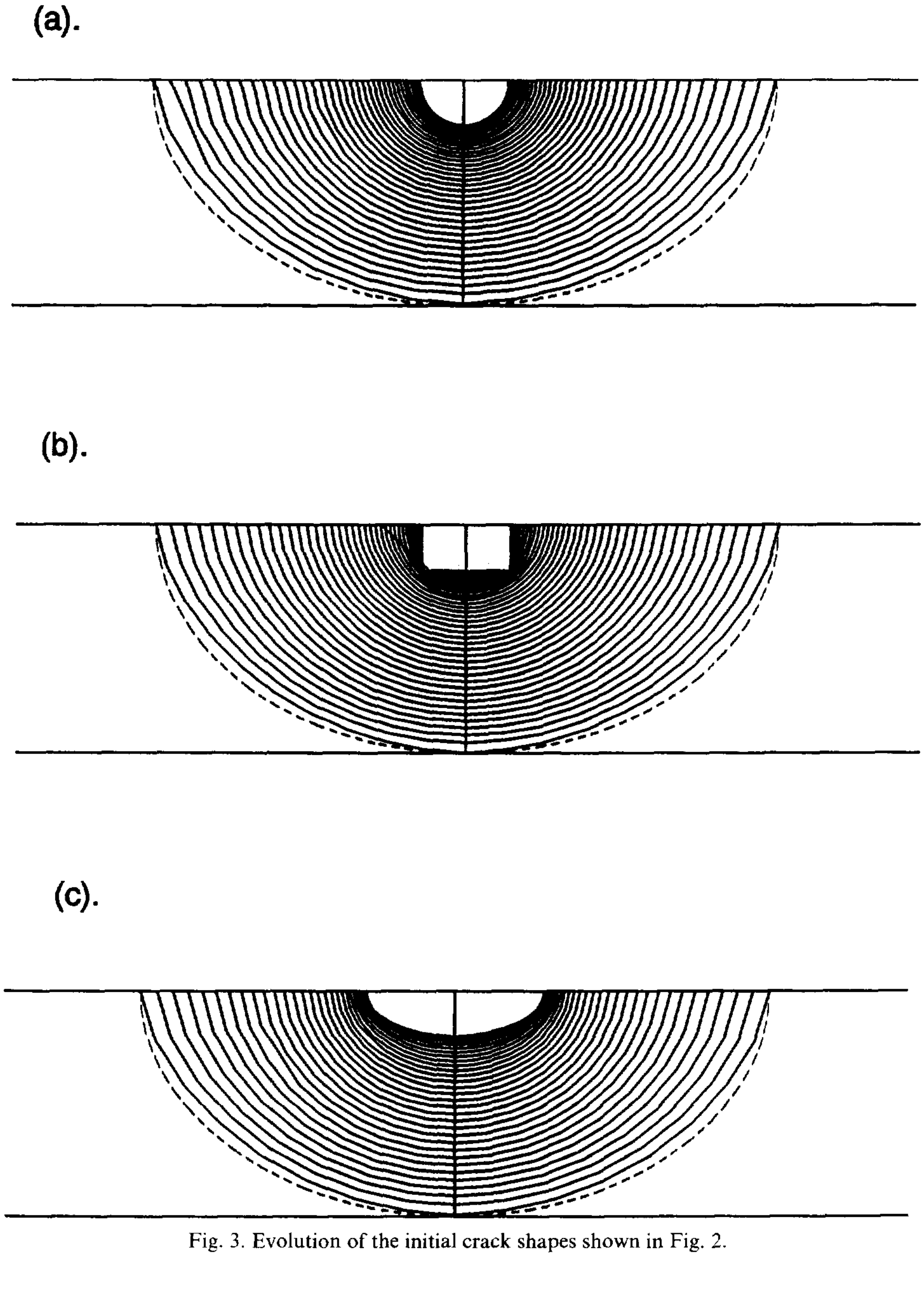

Fig. 3. Evolution of the initial crack shapes shown in Fig. 2.

Fig. 4. However, both experimental observation and our numerical simulation show that the consequence of tensile fatigue will smooth out the initial curvature of the crack front, thus reducing the variation of the stress intensity factor along the front and causing the crack to grow in an approximately self-similar manner. In fact, by the time the cracks have advanced to roughly halfway through the plate, all cracks have attained a similar shape, which gives an approximately constant stress intensity factors along the crack front. Subsequently, all cracks propagate in a similar manner, with the surface of the cracks growing slightly faster than elsewhere. This growth pattern persists until the deepest part of the cracks is very close to the backface of the plate, whereupon the deepest part experiences a sharp increase in stress intensity. The final shapes the cracks take when they hit the backface of the plate are found to be the same. This is also evident from Fig. 4, where the distributions of stress intensity factor along the initial and final crack fronts are plotted.

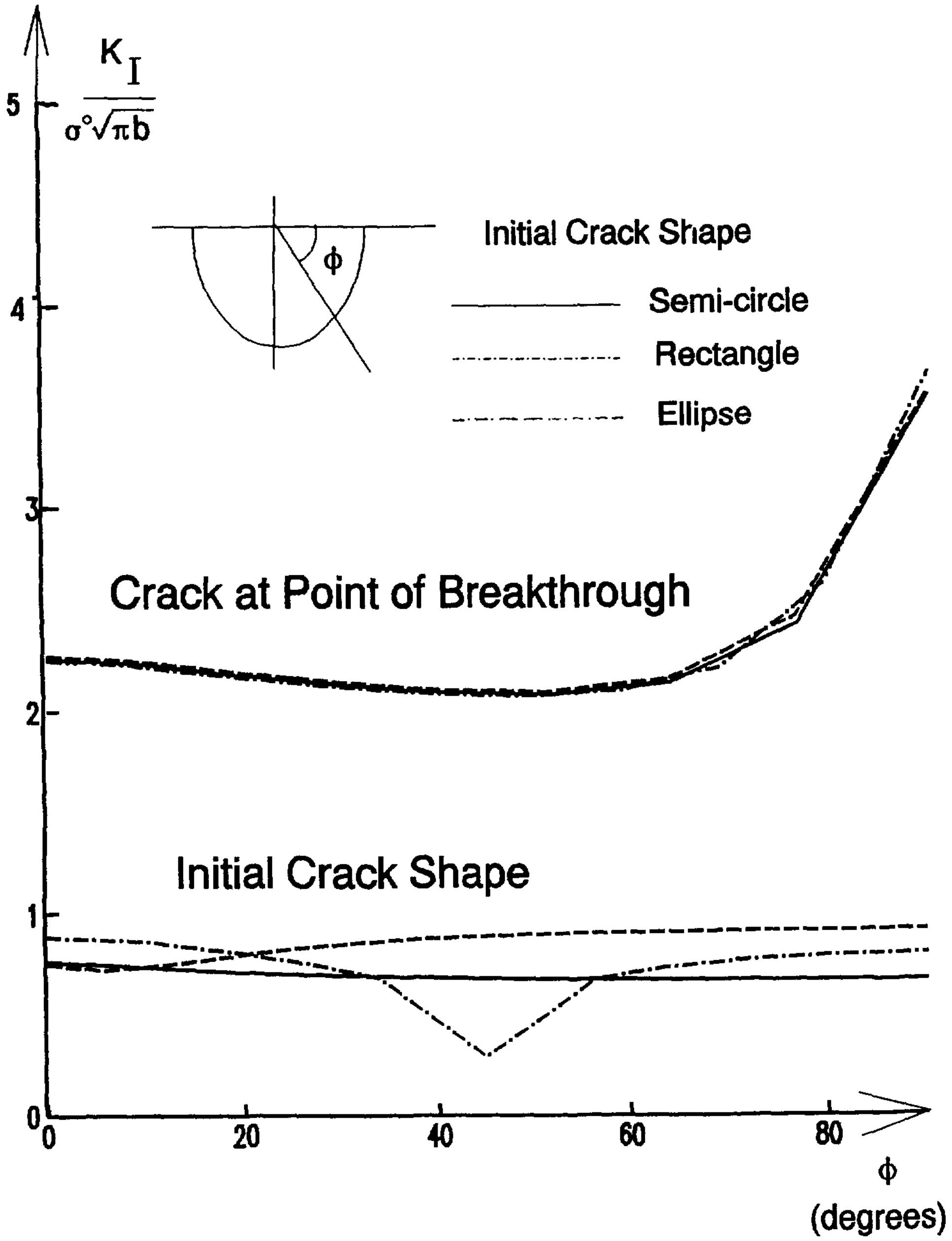

Fig. 4. Variation of crack tip stress intensity factor around crack front for each of the cracks shown in Fig. 3. Values are shown for the initial crack shape, and when the crack is on the point of breaking through to the far surface of the plate.

6. CONCLUSION

We have developed the Green's function and the fundamental dislocation solution for an infinite plate which is free of any surface tractions. It is emphasized that the solutions presented here are quite general in nature, devoid of any kinematic and other assumptions in the classical theories of plates. Further, the use of the image method leads naturally to the isolation of the singularity of the solution, thus facilitating analytical treatment of the singularity in a subsequent analysis of crack problems.

The fundamental dislocation solution has been successfully employed to analyze crack problems in a plate by a continuous distribution of infinitesimal dislocation loops over the crack faces. As demonstrated, a very accurate solution can be achieved using this technique. It is particularly promising for crack growth analysis, as only the crack faces need to be remeshed in a numerical solution, which makes for an efficient computational scheme.

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APPENDIX A. EXPRESSIONS FOR FUNCTIONS $U_1^k(\alpha, z, z'), \ldots$

In the following discussion, we denote that $F(\alpha, z, z') = F^{\beta}(\alpha, z, z')$ $(\beta = 1, 2)$. The functions, S_0, $U_-, \ldots, S_3^3$, appearing in eqns (9) and (15), can therefore be expressed in the forms,

$$U_1^k(\alpha, z, z') = \sum_{i=1}^{4} \{[P_{1i}^k(\alpha) + (1+\alpha z)P_{2i}^k(\alpha)]e^{-(z_i'-z)\alpha}$$

$$+[-P_{3i}^k(\alpha) + (1-\alpha z)P_{4i}^k(\alpha)]e^{-(z_i'+z)\alpha}\}$$

$$U_3^k(\alpha,z,z') = \sum_{i=1}^{4}\{[-P_{1i}^k(\alpha)+(2\kappa_0-\alpha z)P_{2i}^k(\alpha)]e^{-(z_i'-z)\alpha}$$

$$+[-P_{3i}^k(\alpha)-(2\kappa_0+\alpha z)P_{4i}^k(\alpha)]e^{-(z_i'+z)\alpha}\}$$

$$S_0^k(\alpha,z,z') = \frac{1}{2}\sum_{i=1}^{4}\{[P_{1i}^k(\alpha)+(3-2\kappa_0+\alpha z)P_{2i}^k(\alpha)]e^{-(z_i'-z)\alpha}$$

$$+[-P_{3i}^k(\alpha)+(3-2\kappa_0-\alpha z)P_{4i}^k(\alpha)]e^{-(z_i'-z)\alpha}\}$$

$$S_3^k(\alpha,z,z') = \sum_{i=1}^{4}\{[-P_{1i}^k(\alpha)+(\kappa_0-\alpha z)P_{2i}^k(\alpha)]e^{-(z_i'-z)\alpha}$$

$$+[P_{3i}^k(\alpha)+(\kappa_0+\alpha z)P_{4i}^k(\alpha)]e^{-(z_i'+z)\alpha}\}$$

$$S_4^k(\alpha,z,z') = \sum_{i=1}^{4}\{[P_{1i}^k(\alpha)+(1-\kappa_0+\alpha z)P_{2i}^k(\alpha)]e^{-(z_i'-z)\alpha}$$

$$+[P_{3i}^k(\alpha)-(1-\kappa_0-\alpha z)P_{4i}^k(\alpha)]e^{-(z_i'-z)\alpha}\}$$

$$U_2(\alpha,z,z') = 2(1+\kappa_0)\sum_{i=1}^{2}[e^{-(z_i'-z)\alpha}+e^{-(z_i'+z)\alpha}]$$

$$S_5(\alpha,z,z') = (1+\kappa_0)\sum_{i=1}^{2}[e^{-(z_i'-z)\alpha}-e^{-(z_i'+z)\alpha}],$$

$$U_+(\alpha,z,z') = U_1(\alpha,z,z')+\frac{D(\alpha)}{H(\alpha)}U_2(\alpha,z,z')$$

$$U_-(\alpha,z,z') = U_1(\alpha,z,z')-\frac{D(\alpha)}{H(\alpha)}U_2(\alpha,z,z')$$

$$S_+(\alpha,z,z') = S_4(\alpha,z,z')+\frac{D(\alpha)}{H(\alpha)}S_5(\alpha,z,z')$$

$$S_-(\alpha,z,z') = S_4(\alpha,z,z')-\frac{D(\alpha)}{H(\alpha)}S_5(\alpha,z,z'), \tag{A1}$$

where $k = 1, 2, 3$, $\kappa_0 = 1-2\nu$, $z_i' = 2[(i+1)/2]+(-1)^i z'$, with $[\cdot]$ denoting the truncation function, and $H(\alpha) = 1-e^{-2\alpha}$. Further, $P_{ij}^k(\alpha)$ $(k = 1,2,3)$ are polynominals of α, given by

$$P_{11}^k(\alpha) = -\frac{1+c_0^k\kappa_1\kappa_2}{2}+[-2c_0^k d_0^k+\kappa_1 z']\alpha-2(1-z')\alpha^2$$

$$P_{12}^k(\alpha) = -d_0^k-[\kappa_1\kappa_2+c_0^k(1+z')]\alpha-2[\kappa_2+c_0^k\kappa_1 z']\alpha^2-4c_0^k z'\alpha^3$$

$$P_{13}^k(\alpha) = \frac{1+c_0^k\kappa_1\kappa_2}{2}-\kappa_1 z'\alpha$$

$$P_{14}^k(\alpha) = d_0^k+c_0^k z'\alpha$$

$$P_{21}^k(\alpha) = c_0^k\kappa_2+2(1-z')\alpha$$

$$P_{22}^k(\alpha) = c_0^k+2\kappa_2\alpha+4c_0^k z'\alpha^2$$

$$P_{23}^k(\alpha) = -c_0^k\kappa_2+2z'\alpha$$

$$P_{24}^k(\alpha) = -c_0^k$$

$$P_{31}^k(\alpha) = -c_0^k d_0^k-[1-z'+c_0^k\kappa_1\kappa_2]\alpha-2\kappa_1(1-z')\alpha^2$$

$$P_{32}^k(\alpha) = -\frac{1+c_0^k\kappa_1\kappa_2}{2}c_0^k-[2d_0^k+c_0^k\kappa_1 z']\alpha-2[\kappa_1\kappa_2+c_0^k z']\alpha^2-4c_0^k\kappa_1 z'\alpha^3$$

$$P_{33}^k(\alpha) = c_0^k d_0^k-z'\alpha$$

$$P_{34}^k(\alpha) = \frac{1+c_0^k\kappa_1\kappa_2}{2}c_0^k+c_0^k\kappa_1 z'\alpha$$

$$P^k_{41}(\alpha) = -1-2c^k_0\kappa_2\alpha-4(1-z')\alpha^2$$

$$P^k_{42}(\alpha) = -\kappa_2-2c^k_0(1+z')\alpha-4\kappa_2\alpha^2-8c^k_0z'\alpha^3$$

$$P^k_{43}(\alpha) = 1$$

$$P^k_{44}(\alpha) = \kappa_2+2c^k_0z'\alpha \tag{A2}$$

where $\kappa_1 = 1-2\kappa_0$, $\kappa_2 = 1+2\kappa_0$, $c^1_0 = c^2_0 = c^3_0 = -1$, $d^1_0 = d^2_0 = 2\kappa_0$ and $d^3_0 = 1$.